

# CORRECTION TO “TOROIDAL AND KLEIN BOTTLE BOUNDARY SLOPES”

LUIS G. VALDEZ-SÁNCHEZ

ABSTRACT. Let  $M$  be a compact, connected, orientable, irreducible 3-manifold and  $T_0$  an incompressible torus boundary component of  $M$  such that the pair  $(M, T_0)$  is not cabled. In the paper “Toroidal and Klein bottle boundary slopes” [5] by the author it was established that for any  $\mathcal{K}$ -incompressible tori  $F_1, F_2$  in  $(M, T_0)$  which intersect in graphs  $G_{F_i} = F_i \cap F_j \subset F_i$ ,  $\{i, j\} = \{1, 2\}$ , the maximal number of mutually parallel, consecutive, negative edges that may appear in  $G_{F_i}$  is  $n_j + 1$ , where  $n_j = |\partial F_j|$ . In this paper we show that the correct such bound is  $n_j + 2$ , give a partial classification of the pairs  $(M, T_0)$  where the bound  $n_j + 2$  is reached, and show that if  $\Delta(\partial F_1, \partial F_2) \geq 6$  then the bound  $n_j + 2$  cannot be reached; this latter fact allows for the short proof of the classification of the pairs  $(M, T_0)$  with  $M$  a hyperbolic 3-manifold and  $\Delta(\partial F_1, \partial F_2) \geq 6$  to work without change as outlined in [5].

## 1. INTRODUCTION

Let  $M$  be a compact, connected, orientable, irreducible 3-manifold and  $T_0$  an incompressible torus boundary component of  $M$  such that the pair  $(M, T_0)$  is not cabled and *irreducible* (that is,  $M$  is irreducible and  $T_0$  is incompressible in  $M$ ). A punctured torus  $(F, \partial F) \subset (M, T_0)$  is said to be *generated by a (an essential) Klein bottle* if there is a (an essential, resp.) punctured Klein bottle  $(P, \partial P) \subset (M, T_0)$  such that  $F$  is isotopic in  $M$  to the frontier of a regular neighborhood of  $P$  in  $M$ . We also say that  $F$  is  $\mathcal{K}$ -*incompressible* if  $F$  is either incompressible or generated by an essential Klein bottle.

The main purpose of the paper [5] was to establish an upper bound for the maximal number of mutually parallel, consecutive, negative edges that may appear in either graph of intersection  $G_{F_1}, G_{F_2}$  between  $\mathcal{K}$ -incompressible punctured tori  $F_1, F_2$  in  $(M, T_0)$ . In [5, Proposition 3.4] it is proved that for  $\{i, j\} = \{1, 2\}$  and  $n_j = |\partial F_j|$ , if  $G_{F_i}$  contains such a collection of  $n_j + 2$  negative edges then  $M$  is homeomorphic to the trefoil knot exterior or to one of the manifolds  $P \times S^1/[m]$ ,  $m \geq 1$  constructed in [5, §3.4], none of which is a hyperbolic manifold; consequently, if  $M$  is not one of the manifolds listed in [5, Proposition 3.4], the upper bound for such a collection of negative edges was found to be  $n_j + 1$ .

In this paper we show that the list of options for the homeomorphism class of  $M$  given in [5, Proposition 3.4] is incomplete, so that if  $M$  is not one of the manifolds listed in [5, Proposition 3.4] then the correct bound for such families of negative edges in the graph  $G_{F_i}$  is  $n_j + 2$ , and that if the upper bound  $n_j + 2$  is reached then  $(M, T_0)$  belongs to a certain family of examples each of which contains a separating essential twice punctured torus with boundary slope at distance 3 from that of  $F_j$ .

We will use the same notation set up in [5] except that *polarized* will be replaced by *positive* (see Section 2.1); in particular, the tori  $F_1, F_2$  will now be denoted by  $S, T$ , with  $s = |\partial S|$  and  $t = |\partial T|$ . A graph  $G$  in a punctured surface  $F$  is a 1-submanifold properly embedded in  $F$  with vertices the components of  $\partial F$ . The graph  $G$  is *essential* if no edge is parallel into  $\partial F$  and each circle component is essential in  $F$ . The reduced graph  $\overline{G} \subset F$  of  $G$  is the graph obtained from  $G$  by amalgamating each maximal collection of mutually parallel edges  $e_1, \dots, e_k$  of  $G$  into a single arc  $\overline{e} \subset F$ ; we then say that the *size* of  $\overline{e}$  is  $|\overline{e}| = k$ . The symbol  $(+g, b; \alpha_1/\beta_1, \dots, \alpha_k/\beta_k)$  will be used to denote a Seifert fibered manifold over an orientable surface of genus  $g \geq 0$  with  $b \geq 0$  boundary components and  $k \geq 0$  singular fibers of orders  $\beta_1, \dots, \beta_k$ .

The main technical result of this paper is the following.

**Proposition 1.1.** *For each  $t \geq 4$  there is an irreducible pair  $(M_t, T_0)$  with  $\partial M_t = T_0$  which contains properly embedded essential punctured tori  $(S, \partial S)$  and  $(T, \partial T) \subset (M, T_0)$  satisfying the following properties:*

- (1)  $S$  is a separating twice punctured torus and  $T$  is a positive torus with  $|\partial T| = t$ ,
- (2)  $\Delta(\partial S, \partial T) = 3$ ,
- (3)  $S, T$  intersect transversely and minimally in the essential graphs  $G_T = S \cap T \subset T$  shown in Figs. 16 and 17, where the reduced graph  $\overline{G}_S$  consists of 3 negative edges of sizes  $t+2, t, t-2$ ,
- (4)  $M_t(\partial T)$  is a torus bundle over the circle with fiber  $\widehat{T}$ , and  $M_t$  is not homeomorphic to any of the manifolds  $P \times S^1/[m]$  constructed in [5, Proposition 3.4].

It is proved in [6] (preprint, in progress) that each manifold  $M_t$  in Proposition 1.1 is hyperbolic with  $M_t(\partial S) = (+0, 1 : \alpha_1/2, \alpha_2/(t+2)) \cup_{\widehat{S}} (+0, 1 : \alpha_1/2, \alpha_2/(t-2))$ , so  $M_{t_1} \not\approx M_{t_2}$  for  $t_1 \neq t_2$  and  $S$  is generated by a punctured Klein bottle iff  $t = 4$ .

The corrected version of [5, Proposition 3.4] can now be stated as follows.

**Proposition 1.2.** *(Correction to [5, Proposition 3.4]) Let  $(M, T_0)$  be an irreducible pair which is not cabled,  $(T, \partial T) \subset (M, T_0)$  a  $\mathcal{K}$ -incompressible torus with  $t = |\partial T| \geq 1$ , and  $R \subset M$  a surface which intersects  $T$  in essential graphs  $G_R, G_T$ , such that  $G_R$  has at least  $t+2$  mutually parallel, consecutive negative edges. Then  $T$  is a positive torus and one of the following holds:*

- (1) *the conclusion of [5, Proposition 3.4] holds, so  $M$  is homeomorphic to the trefoil knot exterior ( $t = 1$ ) or to one of the manifolds  $P \times S^1/[m]$  ( $t \geq 2$ ),*
- (2)  *$t \geq 4$  and  $(M, T_0)$  is homeomorphic to one of the pairs  $(M_t, T_0)$  of Proposition 1.1, and  $|\overline{e}| \leq t+2$  holds for any edge of the reduced graph  $\overline{G}_R$ ,*
- (3)  *$t = 2$  with  $M = (+0, 1; -1/4, -1/4)$  and  $M(\partial T) = (+0, 0; 1/2, -1/4, -1/4)$ , or  $t = 3$  with  $M = (+0, 1; -1/3, -1/6)$  and  $M(\partial T) = (+0, 0; 1/2, -1/3, -1/6)$ , where in each case the essential annulus  $A \subset (M, T_0)$  satisfies  $\Delta(\partial T, \partial A) = 2$ .*

The smaller bound of  $n_j + 1$  allows for the short proof of the classification of hyperbolic manifolds  $(M, T_0)$  with toroidal or Kleinian Dehn fillings at distance  $6 \leq \Delta \leq 8$  given in [5, §4]. The following result states that in the range  $6 \leq \Delta \leq 8$  the bound  $n_j + 2$  is never reached, which implies that the proofs in [5, §4] work as written.

**Lemma 1.3.** *Suppose  $(M, T_0)$  is a hyperbolic manifold and  $S, T \subset (M, T_0)$  are essential tori such that  $t = |\partial T| \geq 3$  and  $\Delta(\partial S, \partial T) \geq 6$ ; then  $|\bar{e}| \leq t + 1$  holds in  $\bar{G}_S$ .*

Our last lemma summarizes the bounds on the sizes of negative edges of reduced graphs like  $\bar{G}_R$  obtained from the above results.

**Lemma 1.4.** *Let  $(M, T_0)$  be an irreducible pair which is not cabled,  $(T, \partial T) \subset (M, T_0)$  a  $\mathcal{K}$ -incompressible torus with  $t = |\partial T| \geq 1$ , and  $Q \subset M$  a surface which intersects  $T$  in essential graphs  $G_Q = Q \cap T \subset Q$ ,  $G_T = Q \cap T \subset T$ . Then one of the following holds:*

- (1)  $t \leq 3$  and  $M$  is one of the Seifert manifolds  $(+0, 1; 1/2, 1/3)$  ( $t = 1$ ),  $(+0, 1; -1/4, -1/4)$  ( $t = 2$ ), or  $(+0, 1; -1/3, -1/6)$  ( $t = 3$ ),
- (2)  $t \geq 2$  and  $M$  is one of the manifolds  $P \times S^1/[m]$  constructed in [5, Proposition 3.4],
- (3)  $t \geq 4$  and  $M$  is homeomorphic to one of the manifolds  $M_t$  of Proposition 1.1, in which case the bound  $|\bar{e}| \leq t + 2$  holds for all negative edges of  $\bar{G}_Q$ ,
- (4) the bound  $|\bar{e}| \leq t + 1$  holds for all negative edges of  $\bar{G}_Q$ .

In Section 2 we review the notation and constructions given in [5, §2, §3], with which we assume the reader is familiar. In Section 3 we construct the manifolds  $M_t$  for  $t \geq 4$  of Proposition 1.1 and establish the results needed to prove Proposition 1.2 and Lemmas 1.3 and 1.4.

## 2. SLIDABLE AND NON-SLIDABLE BIGONS

**2.1. Generalities.** For any slope  $r$  in  $T_0 \subset \partial M$ ,  $M(r)$  denotes the Dehn filling  $M \cup_{T_0} S^1 \times D^2$ , where  $r$  bounds a disk in the solid torus  $S^1 \times D^2$ . We denote the core of  $S^1 \times D^2$  by  $K_r \subset M(r)$ .

Let  $F$  be a surface properly embedded in  $(M, T_0)$  with  $|\partial F| \geq 1$  and boundary slope  $r$ . Then the surface  $\hat{F} \subset M(r)$  obtained by capping off the components of  $\partial F$  with a disjoint collection of disks in  $S^1 \times D^2$  is a closed surface, which we always assume to intersect  $K_r$  transversely and minimally in  $M(\partial F)$ .

If  $F$  is orientable, we say that  $F$  is *neutral* if  $\hat{F} \cdot K_r = 0$  in  $M(\partial F)$ , where  $\hat{F} \cdot K$  denotes homological intersection number, and that  $F$  is *positive* if  $|\hat{F} \cdot K_r| = |\hat{F} \cap K_r|$ ; the latter is equivalent to the term *polarized* used in [5].

For surfaces  $F_1, F_2 \subset (M, T_0)$  with transverse intersection, for  $i = 1, 2$ ,  $G_{F_i} = F_1 \cap F_2 \subset F_i$  will denote the graph of intersection of  $F_1$  and  $F_2$  in  $F_i$  (with vertices the boundary components of  $F_i$ ).

Following [3], for  $i = 1, 2$  we orient the components of  $\partial F_i$  and coherently on  $T_0$  and say that an edge  $e$  of  $G_{F_i}$  is *positive* or *negative* depending on whether the orientations of the components of  $\partial F_i$  (possibly the same) around a small rectangular regular neighborhood of  $e$  in  $F_i$  appear as in Fig 1.

The following lemma summarizes some of the general properties of graphs of intersection of surfaces in  $(M, T_0)$  that will be relevant in the sequel.

**Lemma 2.1.** *Let  $F_1, F_2$  be properly embedded surfaces in  $(M, T_0)$  with essential graphs of intersection  $G_{F_1} = F_1 \cap F_2 \subset F_1$  and  $G_{F_2} = F_1 \cap F_2 \subset F_2$ .*

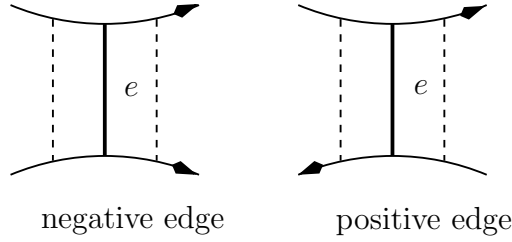


FIGURE 1.

(a) *Parity Rule: for  $\{i, j\} = \{1, 2\}$ , an edge of  $F_1 \cap F_2$  is positive in  $G_{F_i}$  iff it is negative in  $G_{F_j}$  (cf. [3]);*

*moreover, if  $(M, T_0)$  is not cabled,*

(b) *no two edges of  $F_1 \cap F_2$  are parallel in both  $G_{F_1}$  and  $G_{F_2}$  ([1, Lemma 2.1]),*

(c) *if  $F_i$  is a torus and  $G_{F_j}$  has a family of  $n_i + 1$  mutually parallel, consecutive, negative edges then  $F_i$  is a positive torus (cf. [5, Lemma 3.2]).*  $\square$

**2.2. Review of constructions in [5, §2,3].** Suppose  $(M, T_0), T, R$  satisfy the hypothesis in Proposition 1.2, so  $(M, T_0)$  is an irreducible pair with  $T_0$  a torus boundary component of  $M$  which is not cabled,  $T$  a  $\mathcal{K}$ -incompressible torus in  $(M, T_0)$ , and  $R$  any surface properly embedded in  $(M, T_0)$  which intersects  $T$  transversely in essential graphs  $G_T = R \cap T \subset T$  and  $G_R = R \cap T \subset R$ .

We assume there is a collection  $E = \{e_1, e_2, \dots, e_{t+1}, e_{t+2}\}$  of mutually parallel, consecutive, negative edges in  $G_R$ . By Lemma 2.1(c), the torus  $T$  is actually positive and hence incompressible in  $M$ .

The torus  $T$  has orientation vector  $\vec{N}$  shown in Fig. 3, and each vertex of  $T$  is given the orientation induced by  $\vec{N}$  (see Fig. 3). Recall that the vertices of  $G_T, G_R$  are the components of  $\partial T, \partial R$ , respectively; we denote the vertices of  $T$  by  $v_i$ 's and those of  $R$  by  $w_k$ 's. Any two edges  $e, e'$  of  $G_T$  that are incident to the oriented vertex  $v$  split  $v$  into two open subintervals  $(e, e') \subset v$  and  $(e', e) \subset v$ , where  $(e, e')$  is the open subinterval whose *left* and *right* endpoints, as defined by the orientation of  $v$ , come from  $e$  and  $e'$ , respectively.

The collection of edges  $E$  induces a permutation  $\sigma$  of the form  $x \mapsto x + \alpha$  with  $1 \leq \alpha \leq t$ , where  $\gcd(t, \alpha) = 1$  by [5, Lemma 3.2]; the definition of  $\sigma$  requires that a common orientation be given to the edges of  $E$ , and reversing the orientation of such edges replaces  $\sigma$  with its inverse, hence  $\alpha$  with  $t - \alpha$ .

Cutting  $M$  along  $T$  produces an irreducible manifold  $M_T = \text{cl}(M \setminus N(T))$  with copies  $T^1, T^2 \subset \partial M_T$  of  $T$  on its boundary and strings  $I'_{1,2}, I'_{2,3}, \dots, I'_{t,1} \subset \partial M_T$  such that  $M_T/\psi = M$  for some orientation preserving homeomorphism  $\psi : T^1 \rightarrow T^2$ , where  $T^1, T^2$  are oriented by normal vectors  $\vec{N}^1, \vec{N}^2$  as shown in Fig. 4.

We assume that the edges of  $E$  and  $T^1 \cap R, T^2 \cap R$  are arranged in  $G_R$  as shown in Fig. 2. Each edge  $e_i$  of  $E$  and vertex  $v_i$  of  $T$  gives rise to two copies of itself  $e_i^1, v_i^1 \subset T^1$  and  $e_i^2, v_i^2 \subset T^2$  such that  $\psi(e_i^1) = e_i^2$  and  $\psi(v_i^1) = v_i^2$ .

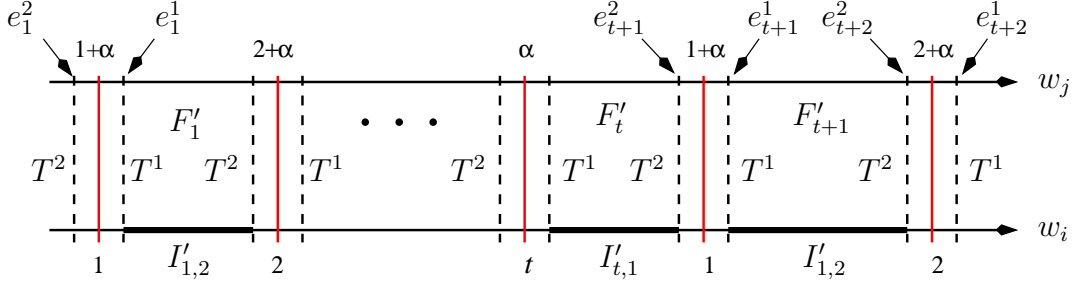


FIGURE 2.

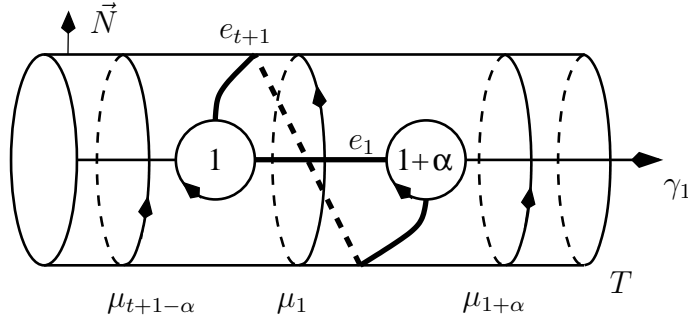


FIGURE 3.

The collection  $E$  also gives rise to two essential cycles in  $T$ ,  $\gamma_1 = e_1 \cup e_2 \cup \dots \cup e_t$  and  $\gamma_2 = e_2 \cup e_3 \cup \dots \cup e_t \cup e_{t+1}$ , such that  $\Delta(\gamma_1, \gamma_2) = 1$  holds in  $\hat{T}$ . The edges  $e_1^1 \cup e_2^1 \cup \dots \cup e_t^1$  form the essential cycle  $\gamma_1^1$  in  $T^1$ , while  $e_2^2 \cup e_3^2 \cup \dots \cup e_{t+1}^2$  form the essential cycle  $\gamma_2^2$  in  $T^2$ , such that the bigon faces  $F'_1, F'_2, \dots, F'_t$  bounded by  $e_1, e_2, \dots, e_{t+1}$  in  $R \cap M_T$  form an essential annulus  $\mathcal{A}$  in  $M_T(\partial T)$  as shown in Fig. 4. For simplicity, we refer to the union of the bigons  $F'_1, F'_2, \dots, F'_t$  in  $M_T$  also as the annulus  $\mathcal{A}$ .

The next result describes the embedding of  $\mathcal{A} \cup F'_{t+1}$  in  $M_T$  and the structures of  $M_T$  and  $M_T(\partial T)$ ; its proof follows immediately from the arguments of [5, Lemma 3.3].

**Lemma 2.2.** ([5, Lemma 3.3]) *Up to homeomorphism, the bigons  $F'_1, \dots, F'_{t+1}$  lie in  $M_T$  as shown in Fig. 4. In particular,  $M_T \approx T \times I$  is a genus  $t+1$  handlebody with  $F'_1, \dots, F'_{t+1}$  a complete disk system,  $\partial M = T_0$ , and  $M(\partial T)$  is a torus bundle over the circle with fiber  $\hat{T}$ .  $\square$*

Finally, we construct auxiliary circles  $\mu_i$  in  $T$  having the same slope in  $\hat{T}$  as the cycle  $e_1 \cup e_{t+1}$ , oriented and labeled as shown in Fig. 3. The counterparts  $\mu_i^1 \subset T^1$  of the  $\mu_i$ 's are shown in Fig. 4, while the circles  $\mu_i^2 \subset T^2$  are represented abstractly in Fig. 4 since the location of the edge  $e_1^2 \subset T^2$  is not given yet.

**2.3. Review of the argument of [5, Proposition 3.4].** At this point the argument used in the proof of [5, Lemma 3.6] states that, for  $t \geq 2$ ,

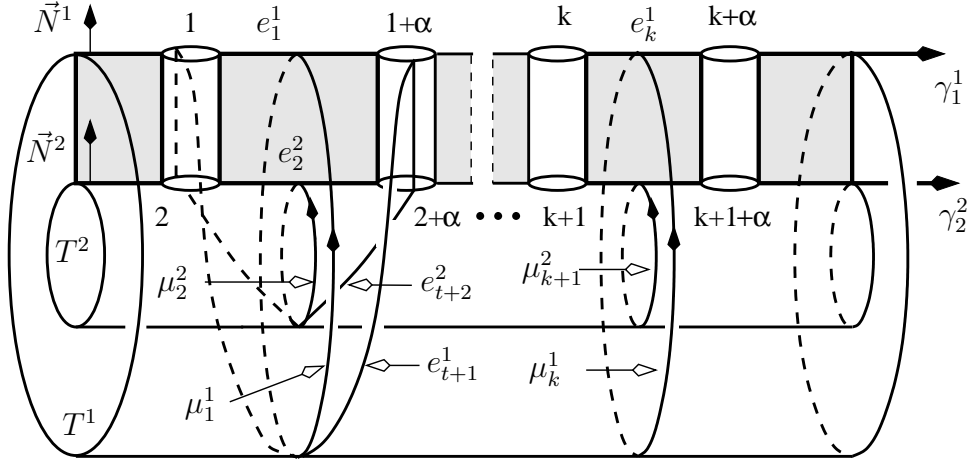


FIGURE 4.

...the faces  $F'_1$  and  $F'_{t+1}$  can be isotoped in  $M_T$  to construct an annulus  $A_1 \subset M_T$  with boundary the circles  $\mu_1^1 \cup \mu_2^2$ , which under their given orientations remain coherently oriented relative to  $A_1$ . Via the product structure  $M_T = T \times I$ , it is not hard to see that each pair of circles  $\mu_k^1, \mu_{k+1}^2$  cobounds such an annulus  $A_k \subset M_T$  for  $1 \leq k \leq t$ , with the oriented circles  $\mu_k^1, \mu_{k+1}^2$  coherently oriented relative to  $A_k$ ; these annuli  $A_k$  can be taken to be mutually disjoint and  $I$ -fibered in  $M_T = T \times I$ . Since  $\psi(\mu_k^1) = \mu_k^2$  (preserving orientations), the union  $A_1 \cup A_2 \cdots \cup A_t$  yields a closed nonseparating torus  $T''$  in  $M$ , on which the circles  $\mu_1, \mu_2, \dots, \mu_t$  appear consecutively in this order and coherently oriented.

The problem with the above argument is that it is assumed from the beginning that the boundary of the annulus  $A_1$  must necessarily be  $\mu_1^1 \cup \mu_2^2$ , which, as we shall see next, is not the case and leads to the present correction to [5, Proposition 3.4].

The isotopy of  $F'_1$  and  $F'_{t+1}$  in  $M_T$  mentioned in the above quote can be thought of as the result of a *sliding process*, where the corners of the face  $F'_{t+1}$  are slid onto the face  $F'_1$  so as to coincide with each other, at which point the isotoped face  $F'_{t+1}$  becomes the annulus  $A_1$  properly embedded in  $M_T$  with boundary the circles  $\mu_1^1 \cup \mu_2^2$ .

We will say that the bigon  $F'_{t+1}$  is *slidable (relative to the annulus  $\mathcal{A}$ )* whenever the annulus  $A_1$  produced by the above isotopy of  $F'_{t+1}$  satisfies  $\partial A_1 = \mu_1^1 \cup \mu_2^2$ , and otherwise that  $F'_{t+1}$  is *non-slidable*. Equivalently,  $F'_{t+1}$  is slidable iff the cycles  $\psi(e_1^1 \cup e_{t+1}^1) = e_1^2 \cup e_{t+1}^2 \subset T^2$  and  $e_2^2 \cup e_{t+2}^2 \subset T^2$  have the same slope in  $\widehat{T}^2$ , that is iff the cycles  $e_1 \cup e_{t+1}$  and  $e_2 \cup e_{t+2}$  have the same slope in  $\widehat{T}$ .

As we shall see in the next section, there are two combinatorially different embeddings of the edge  $e_1^2$  in  $T^2$  which correspond to the annulus  $A_1$  being slidable or not; the generic embeddings  $e_1^2 \subset T^2$  are shown in Fig. 11, the slidable case which produces the circles  $\mu_i^2 \subset T^2$  shown in Fig. 4), and Fig. 12, the non-slidable case.

Using this notation we summarize [5, Proposition 3.4] as follows:

**Lemma 2.3.** [5, Proposition 3.4]

- (1) If  $t = 1$  then  $M$  is the exterior of the trefoil knot  $(+0, 1; 1/2, 1/3)$ .
- (2) If  $t \geq 2$  and  $F'_{t+1}$  is slidable then  $M$  is homeomorphic to one of the manifolds  $P \times S^1/[m]$  constructed in [5, §3.4], in which case  $M$  is not Seifert fibered and contains a closed nonseparating torus.  $\square$

We shall see below that in most cases, which include those with  $t \geq 4$  and  $\alpha \not\equiv \pm 1 \pmod t$ , the bigon  $F'_{t+1}$  is slidable, and that the exceptions with  $t \geq 4$  form the family  $M_t$  of Proposition 1.1.

### 3. MAIN RESULTS

In this section we assume that  $(M, T_0), T, R$  satisfy the hypothesis of Proposition 1.2; also, the mutually parallel edges  $E = \{e_1, e_2, \dots, e_{t+1}, e_{t+2}\}$  in  $G_R$  are labeled as in Fig. 2, induce the permutation  $\sigma(x) \equiv x + \alpha \pmod t$  with  $\gcd(t, \alpha) = 1$ , and cobound bigon faces  $F'_1, \dots, F'_{t+1}$  embedded in  $M_T$  as shown in Fig. 4. The case  $t = 1$  is considered in Lemma 2.3(1).

#### 3.1. The cases $t = 2, 3$ .

**Lemma 3.1.** *If  $t = 2, 3$  then  $(M, T_0)$  satisfies the conclusion of [5, Proposition 3.4] or of Proposition 1.2(3).*

*Proof.* For  $t = 2$  we must have  $\alpha = 1$ ; for  $t = 3$  we may also assume that  $\alpha = 1$  after reversing the orientation of the edges of  $E$  if necessary.

We begin with a detailed analysis of the case  $t = 2$ . Fig. 5(a) shows the edges  $e_i$  and bigons  $F'_i$  of  $E$  in  $G_R$ . By Lemma 2.2, up to homeomorphism, the annulus  $\mathcal{A}$  cobounded by the cycles  $\gamma_1^1 = e_1^1 \cup e_2^1 \subset T^1$  and  $\gamma_2^2 = e_2^2 \cup e_3^2 \subset T^2$  and the bigon  $F'_3$  cobounded by  $e_3, e_4$  may be assumed to lie in  $M_T$  as shown in Fig. 5(b) or (c); for simplicity, the upper labels in the edges will not be shown in the figures representing  $M_T$ .

Notice that the embeddings of the edges  $e_2$  and  $e_3$  are determined in both  $T^1$  and  $T^2$  at this point, but that the embeddings of  $e_1$  and  $e_4$  are so far determined only in  $T^1$  and  $T^2$ , respectively.

To determine the possible embeddings of  $e_1^2$  in  $T^2$  we observe that in either Fig. 5(b),(c) the endpoint of  $e_1^1$  in  $v_1^1$  lies in the interval  $(e_3^1, e_2^1)$ , and that the same statement holds for the endpoint of  $e_1^1$  in  $v_2^1$ . Since  $\psi$  maps each  $v_i^1 \subset T^1$  to  $v_i^2 \subset T^2$  and each  $e_i^1 \subset T^1$  to  $e_i^2 \subset T^2$ , the endpoints of  $e_1^2$  in  $v_1^2, v_2^2$  must also lie in the corresponding intervals  $(e_3^2, e_2^2)$ . As the edge  $e_4^2$  is already embedded in  $T^2$  and its endpoint on  $v_2^2$  lies in  $(e_3^2, e_2^2)$ , it follows that the endpoint of  $e_1^2$  in  $v_2^2$  must lie in one of the intervals  $(e_3^2, e_4^2)$  or  $(e_4^2, e_2^2)$ .

The first option is represented in Fig. 5(b); the placement of  $e_1^2$  in  $T^2$  is then uniquely determined using the fact that no two edges of  $E$  are mutually parallel in  $T$ . Since the cycles  $e_1^2 \cup e_3^2$  and  $e_2^2 \cup e_4^2$  have the same slope in  $\widehat{T}^2$ , the bigon  $F'_3$  is slidable and so by Lemma 2.3 the

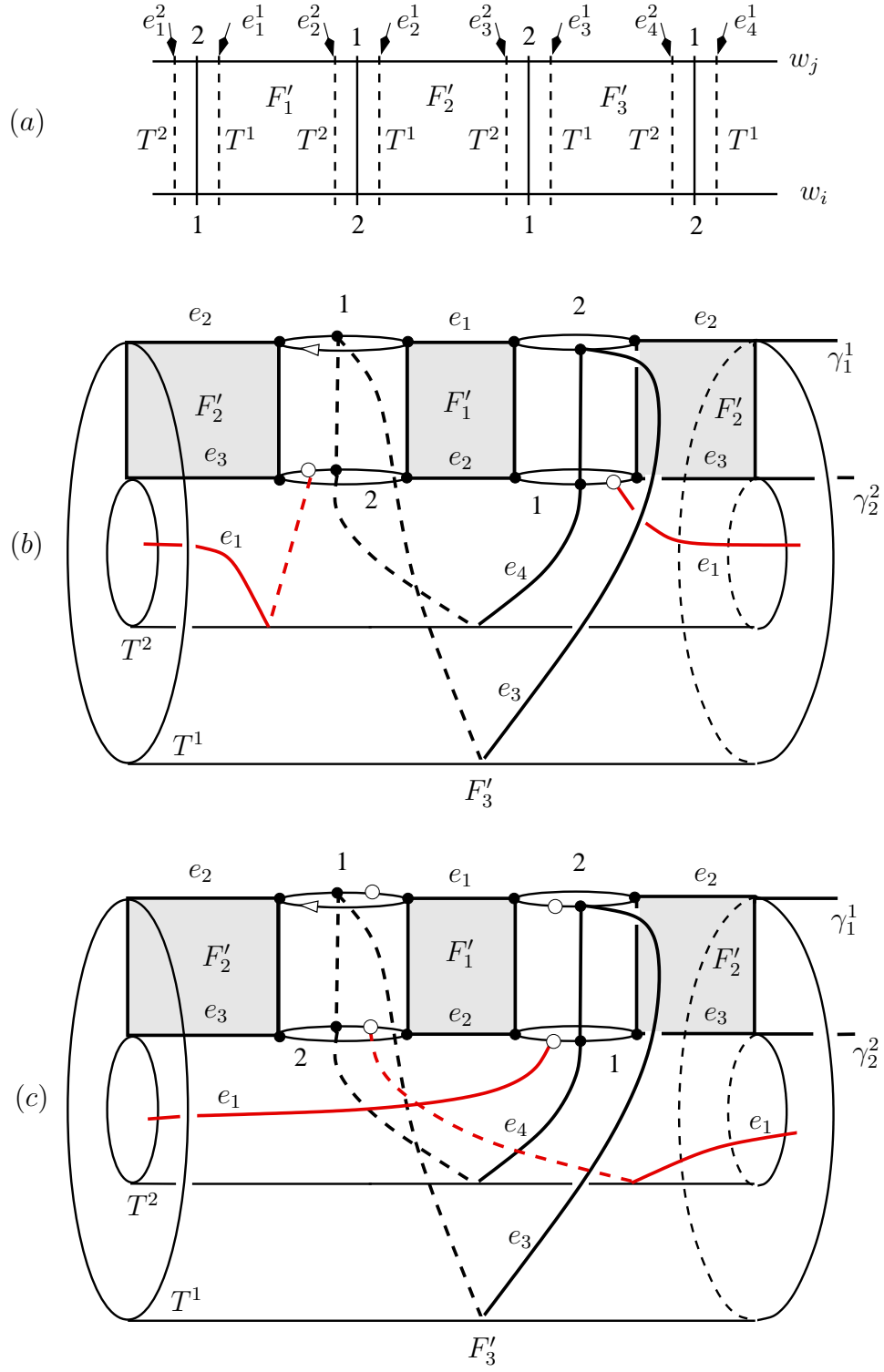


FIGURE 5.



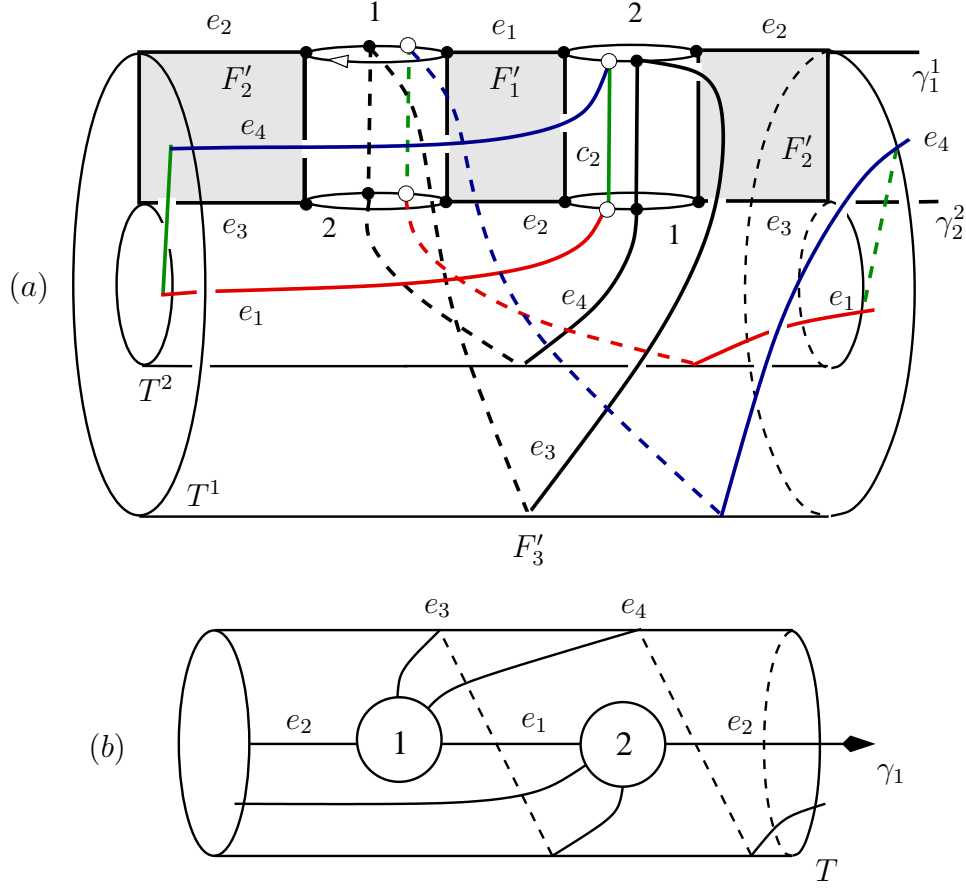


FIGURE 6.

pair  $(M, T_0)$  satisfies the conclusion of [5, Proposition 3.4]. The embedding of  $e_4^1$  in  $T^1$  is now uniquely determined as well: its endpoints lie in the intervals  $(e_1^1, e_2^1) \subset v_1^1$  and  $(e_1^1, e_2^1) \subset v_2^1$ .

The second option is represented in Fig. 5(c), and here we have that  $\Delta(e_1^2 \cup e_3^2, e_2^2 \cup e_4^2) = 2 \neq 0$  so  $F'_3$  is not slidable. The endpoints of  $e_4^2$  in both vertices  $v_1^2, v_2^2$  are now located in the intervals  $(e_3^2, e_1^2)$ , so the endpoints of  $e_4^1$  in both vertices  $v_1^1, v_2^1$  must also be located in the intervals  $(e_3^1, e_1^1)$ ; the endpoints of  $e_4^1$  are indicated in Fig. 5(c) by open circles in  $v_1^1, v_2^1$ . The only possible embedding of the edge  $e_4^1$  in  $T^1$  is shown in Fig. 6(a).

Now, by Lemma 2.2 the bigons  $F'_1, F'_2, F'_3$  form a complete disk system of the handlebody  $M_T$ . Observe that the endpoints of  $e_4^1$  and  $e_2^1$  can be connected via arcs  $c_1, c_2$  along the strings  $I'_{1,2}, I'_{2,1}$  that are disjoint from the corners of the bigons  $F'_1, F'_2, F'_3$ . It follows that the circle  $e_4^1 \cup e_2^1 \cup c_1 \cup c_2 \subset \partial M_T$  is disjoint from  $F'_1, F'_2, F'_3$  and hence bounds a disk  $F'_4$  in  $M_T$  disjoint from  $F'_1, F'_2, F'_3$ . It is not hard to see that the quotient  $A = (F'_1 \cup F'_2 \cup F'_3 \cup F'_4)/\psi$  is a surface in  $(M, T_0)$  which intersects  $T$  transversely and minimally with  $\Delta(\partial T, \partial A) = 2$  and  $G_{T,A} = T \cap A \subset T$  the essential graph of Fig. 6(b). Since  $T$  is positive, by the parity rule  $A$  must be a neutral annulus, hence each face of  $G_{T,A}$  is necessarily a Scharlemann cycle of length 4, and since  $M$  is irreducible it follows that  $A$  must separate  $M$  into two

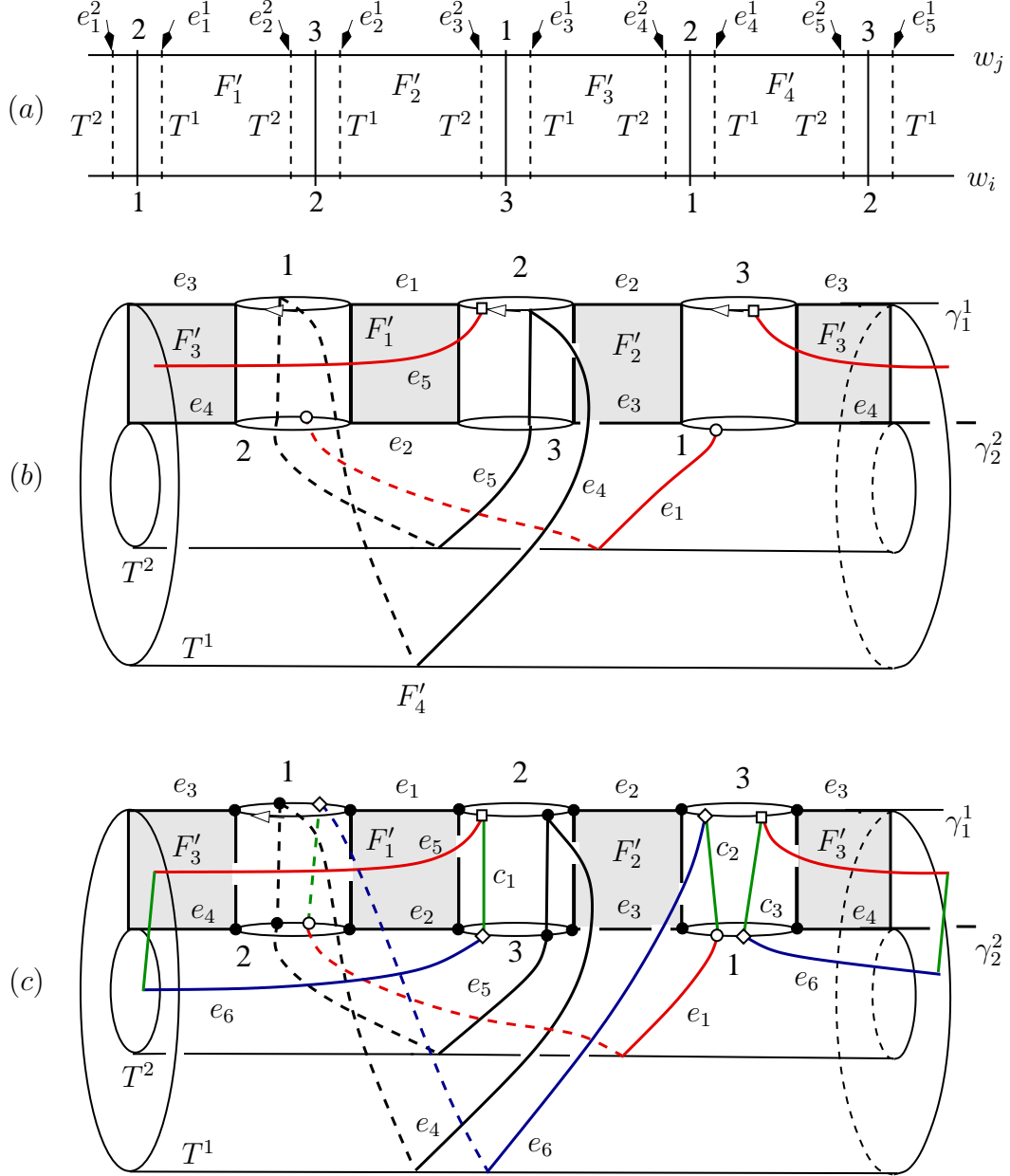


FIGURE 7.

solid tori whose meridian disks, that is the faces of  $G_{T,A}$ , each intersect  $A$  coherently in 4 arcs. Therefore  $M$  is a Seifert fibered manifold over the disk with two singular fibers of indices 4, 4, so  $M(\partial T)$  is a Seifert fibered torus bundle over the circle with horizontal bundle fiber  $\hat{T}$ . By the classification of such torus bundles (cf [2, §2.2]), it follows that  $M(\partial T) = (+0, 0; 1/2, -1/4, -1/4)$  and hence that  $M = (+0, 1; -1/4, -1/4)$ .

The case  $t = 3$  is handled in a similar way: Fig. 7(a) shows the labeling of the edges  $E = \{e_1, \dots, e_5\}$  and the bigon faces  $F'_1, \dots, F'_4$  they cobound in  $G_R$ , while Fig. 7(b) shows

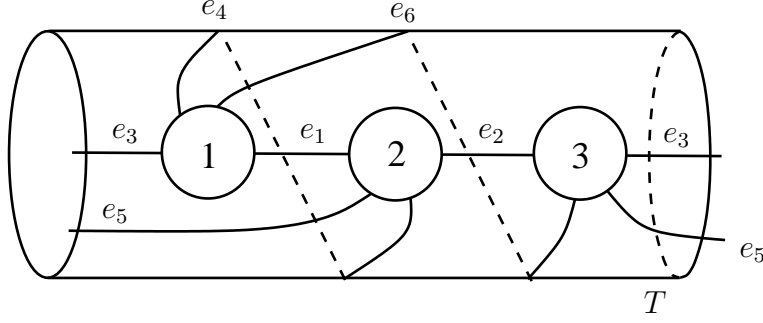


FIGURE 8.

the embedding of the bigons  $F'_1, \dots, F'_4$  in  $M_T$ , up to homeomorphism. Again there are only two possible embeddings for the edge  $e_1^2$  in  $T^2$  depending on whether the endpoint of  $e_1^2$  in  $v_2^2$  lies in  $(e_4^2, e_5^2)$  or  $(e_5^2, e_2^2)$ . In the first case the bigon  $F'_4$  is again slidable; we now sketch the details for the case corresponding to the later option, where  $F'_4$  is non-slidable.

Fig. 7(b) shows the embedding of  $e_1^2 \subset T^2$  which makes the bigon  $F'_4$  non-slidable and forces the embedding of  $e_5^1 \subset T^1$ . At this point all the edges in  $E^2 = \{e_1^2, \dots, e_5^2\}$  have been embedded in  $T^2$ . Fig. 7(c) shows the embedding of a 6th edge  $e_6^2$  in  $T^2$  with endpoints on the intervals  $(e_4^2, e_1^2) \subset v_1^2$  and  $(e_4^2, e_1^2) \subset v_1^2$  and which is disjoint from and not parallel in  $T^2$  to any of the edges  $e_1^2, e_2^2, \dots, e_5^2$ . These properties imply that  $e_6^1 = \psi^{-1}(e_6^2)$  must be the edge in  $T^1$  sketched in Fig. 7(c).

As before, it is possible to connect the endpoints of  $e_5^1$  and  $e_6^2$  via corners  $c_1 \subset I'_{2,3}$  and  $c_2 \subset I'_{3,1}$ , and the endpoints of  $e_6^1$  and  $e_1^2$  via corners  $c_3 \subset I'_{3,1}$  and  $c_4 \subset I'_{1,2}$ , so that the circles  $e_5^1 \cup e_6^2 \cup c_1 \cup c_2 \subset \partial M_T$  and  $e_6^1 \cup e_1^2 \cup c_3 \cup c_4 \subset \partial M_T$  bound disks  $F'_5 \subset M_T$  and  $F'_6 \subset M_T$ , respectively, with the property that the enlarged collection of disks  $F'_1, \dots, F'_6$  is disjoint (see Fig. 7(c)).

It follows that  $A = (F'_1 \cup F'_2 \cup \dots \cup F'_6) / \psi$  is a neutral annulus in  $(M, T_0)$  which intersects  $T$  transversely and minimally with  $G_{T,A}$  the essential graph of Fig. 8 and  $\Delta(\partial T, \partial A) = 2$ . Since  $|\partial A| = 2$ , each face of  $G_{T,A}$  is necessarily a Scharlemann cycle and hence  $A$  must separate  $M$  into two solid tori with meridian disks the faces of  $G_{T,A}$ . As  $G_{T,A}$  has two trigon faces on one side of  $A$  and a 6-sided face in the other side of  $A$ , we must have that  $M$  is a Seifert manifold over the disk with two singular fibers of indices 3, 6, which by the classification of such torus bundles (cf [2, §2.2]) implies that  $M(\partial T) = (+0, 0; 1/2, -1/3, -1/6)$  and hence that  $M = (+0, 1; -1/3, -1/6)$ . The lemma follows.  $\square$

**3.2. The generic case  $t \geq 4$ .** Here we assume that  $t \geq 4$ ; we first show that for most values of  $\alpha$  the bigon  $F'_{t+1}$  is slidable.

**Lemma 3.2.** *If  $\alpha \not\equiv \pm 1 \pmod t$  then the bigon  $F'_{t+1}$  is slidable.*

*Proof.* If  $t = 4$  then  $\alpha \equiv \pm 1 \pmod t$ . For  $t \geq 5$  the condition  $\alpha \not\equiv \pm 1 \pmod t$  is equivalent to saying that the pairs of strings  $\{I'_{1,2}, I'_{1+\alpha, 2+\alpha}\}$  and  $\{I'_{2,3}, I'_{2+\alpha, 3+\alpha}\}$  are disjoint. These four

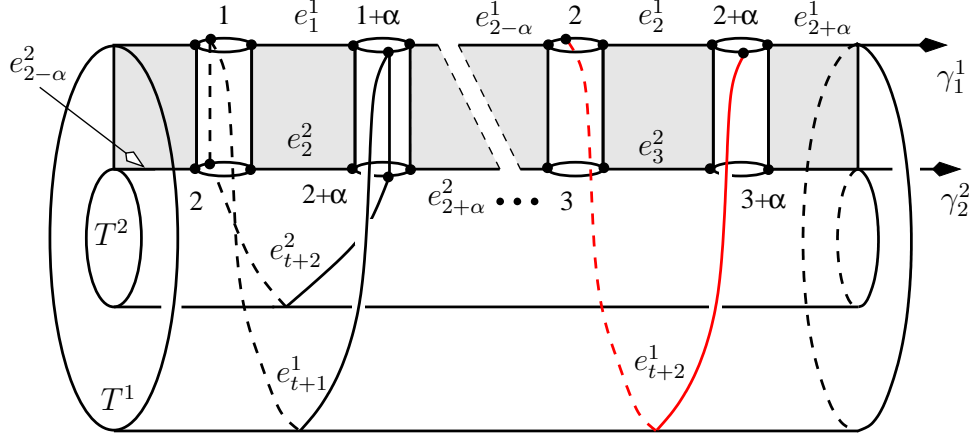


FIGURE 9.

strings are shown in Fig. 9 along with the embeddings of the faces  $F'_1, \dots, F'_{t+1}$  from Fig. 4. As  $e_{t+2}^2$  has one endpoint on the interval  $(e_{2-\alpha}^2, e_2^2) \subset v_2^2$  and the other on  $(e_{2+\alpha}^2, e_2^2) \subset v_{2+\alpha}^2$ , the endpoints of  $e_{t+2}^1 = \psi^{-1}(e_{t+2}^2)$  must lie on  $(e_{2-\alpha}^1, e_2^1) \subset v_2^1$  and  $(e_{2+\alpha}^1, e_2^1) \subset v_{2+\alpha}^1$ . Therefore the edge  $e_{t+2}^1$  must be embedded in  $T^1$  as shown in Fig. 9, which implies that the cycles  $e_1 \cup e_{t+1}$  and  $e_2 \cup e_{t+2}$  have the same slope in  $\hat{T}$  and hence that  $F'_{t+1}$  is slidable.  $\square$

From here on we assume that  $\alpha \equiv \pm 1 \pmod{t}$ , and after reversing the orientation of the edges of  $E$ , if necessary, we will take  $\alpha = 1$ .

**Lemma 3.3.** *If  $\alpha = 1$  and  $t \geq 4$  then, up to homeomorphism, the bigons and edges of  $E$  are embedded in  $M_T$  and  $T$ , respectively, as shown in Fig. 11(a),(b) if  $F'_{t+1}$  is slidable, or Fig. 12(a),(b) if  $F'_{t+1}$  is non-slidable. In the latter case, if  $\bar{e} \subset \bar{G}_R$  is the negative edge that contains the edges of  $E$  then  $\bar{e} = E$ , so  $|\bar{e}| = t + 2$ .*

*Proof.* By the argument of [5, Lemma 3.6] (see in particular [5, Fig. 11] with  $\alpha = 1$ ) we may assume that, up to homeomorphism, the bigons of  $E$  are embedded in  $M_T$  as shown in Fig. 10, where  $t \geq 4$ . Notice that the embedding of the bigons of  $E$  in  $M_T$  only determines the embeddings of the edges  $e_1, e_2, \dots, e_{t+1}$  in  $T^1$  and of  $e_2, e_3, \dots, e_{t+2}$  in  $T^2$ . To determine the possible embeddings of  $e_1$  in  $T^2$  and  $e_{t+2}$  in  $T^1$  we proceed as follows.

From Fig. 10, the endpoints of  $e_1^1$  in  $T^1$  are located in the intervals  $(e_{t+1}^1, e_t^1) \subset v_1^1$  and  $(e_{t+1}^1, e_2^1) \subset v_2^1$ ; since  $\psi(T^1) = T^2$ , it follows that the endpoints of  $e_1^2$  in  $T^2$  must be located in the intervals  $(e_{t+1}^2, e_t^2) \subset v_1^2$  and  $(e_{t+1}^2, e_2^2) \subset v_2^2$ .

The interval  $(e_{t+1}^2, e_2^2) \subset v_2^2$  is split into the two subintervals  $(e_{t+1}^2, e_{t+2}^2)$  and  $(e_{t+2}^2, e_2^2)$  by the endpoint of  $e_{t+2}^2$  in  $v_2^2$ , which gives rise to two possible locations of the endpoint of  $e_1^2$  in  $v_2^2$ . The endpoints of  $e_1^2$  are denoted by open circles in  $v_1^2, v_2^2 \subset T^2$  in Fig. 10. Since all faces of the graphs on  $T^1, T^2$  of Fig. 10 are disks, the embedding of  $e_1^2$  in  $T^2$  is completely determined by the location of its endpoints; therefore there are exactly two possible embeddings of  $e_1^2$  in  $T^2$ .

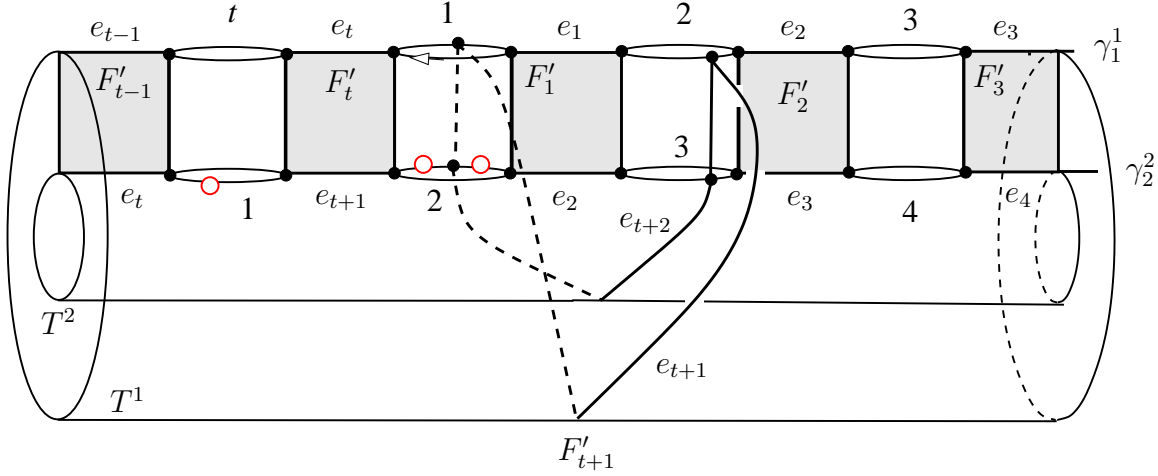


FIGURE 10.

**Case 1.** The endpoint of  $e_1^2$  in  $v_2^2$  lies in  $(e_{t+1}^2, e_{t+2}^2)$ .

This case is represented in Fig. 11(a). Now the endpoints of  $e_{t+2}^2$  in  $T^2$  lie in the intervals  $(e_1^2, e_2^2) \subset v_2^2$  and  $(e_3^2, e_4^2) \subset v_3^2$ , and so the endpoints of  $e_{t+2}^1 = \psi^{-1}(e_{t+2}^2)$  in  $T^1$  must lie in  $(e_1^1, e_2^1) \subset v_2^1$  and  $(e_3^1, e_4^1) \subset v_3^1$ . Therefore  $e_{t+2}^1$  must be the edge in  $T^1$  shown in Fig. 11(a). The graph in  $T$  produced by the edges of  $E$  is shown in Fig. 11(b); as the cycles  $e_1 \cup e_{t+1}$  and  $e_2 \cup e_{t+2}$  have the same slope in  $\widehat{T}$ , the bigon  $F'_{t+1}$  is slidable.

**Case 2.** The endpoint of  $e_1^2$  in  $v_2^2$  lies in  $(e_{t+2}^2, e_2^2)$ .

The embedding of  $e_1^2$  in  $T^2$  is shown in Fig. 12(a). The endpoints of  $e_{t+2}^2$  in  $T^2$  lie in the intervals  $(e_{t+1}^2, e_1^2) \subset v_2^2$  and  $(e_3^2, e_4^2) \subset v_3^2$ , so the endpoints of  $e_{t+2}^1$  in  $T^1$  lie in  $(e_{t+1}^1, e_1^1) \subset v_2^1$  and  $(e_3^1, e_4^1) \subset v_3^1$ ; thus  $e_{t+2}^1$  must lie in  $T^1$  as shown in Fig. 12(a). The graph in  $T$  produced by the edges of  $E$  is shown in Fig. 12(b); this time the cycles  $e_1 \cup e_{t+1}$  and  $e_2 \cup e_{t+2}$  have slopes in  $\widehat{T}$  at distance 1 and so the bigon  $F'_{t+1}$  is nonslidable.

Suppose there is a negative edge  $e_0 \subset \bar{e} \setminus E$  in  $G_R$  which cobounds a bigon  $F_0$  with  $e_1$ , so that  $F_0' \subset M_T$  is cobounded by the edges  $e_0^1 \subset T^1$  and  $e_1^2 \subset T^2$ . Since  $e_1^2$  has endpoints in  $(e_{t+1}^2, e_t^2) \subset v_1^2$  and  $(e_{t+2}^2, e_2^2) \subset v_2^2$ , the corners of  $F_0'$  must lie in the strings  $I'_{t,1}$  and  $I'_{1,2}$  and so the endpoints of  $e_0^1$ , represented by squares in Fig. 12, necessarily must lie in  $(e_t^1, e_{t-1}^1) \subset v_t^1$  and  $(e_{t+1}^1, e_1^1) \subset v_1^1$ . This is impossible since such endpoints are separated by the edge  $e_{t+2} \subset T$  (see Fig. 12(a),(b)), so no such edge  $e_0$  exists in  $\bar{e}$ . A similar argument shows that  $\bar{e} \setminus E$  does not contain any edge adjacent to  $e_{t+2}$ . Therefore  $E = \bar{e}$  and so  $|\bar{e}| = t + 2$ .  $\square$

**Corollary 3.4.** If  $t \geq 4$  and  $M$  is not homeomorphic to any of the manifolds  $P \times S^1/[m]$  then  $|\bar{e}| \leq t + 2$  holds for any negative edge  $\bar{e}$  of  $\bar{G}_R$ .

*Proof.* Let  $\bar{e}$  be any negative edge of  $\bar{G}_R$  with  $|\bar{e}| \geq t + 2$ , and let  $E = \{e_1, \dots, e_{t+2}\}$  be any collection of  $t + 2$  consecutive edges in  $\bar{e}$ . Since  $M$  is not any of the manifolds  $P \times S^1/[m]$ ,

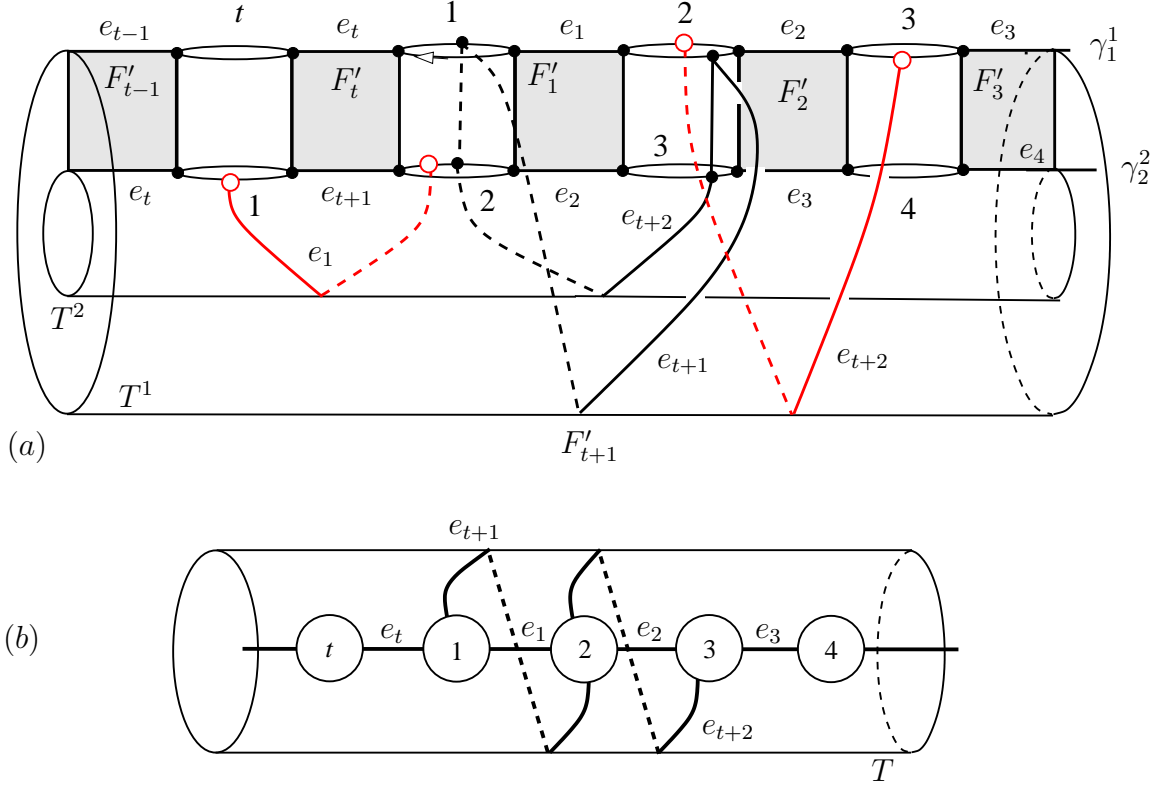


FIGURE 11.

by Lemmas 2.3 and 3.2 the permutation induced by  $E$  must be of the form  $x \rightarrow x \pm 1$ , and hence  $|\bar{e}| = t + 2$  holds by Lemma 3.3.  $\square$

We now prove Lemma 1.3:

**Proof of Lemma 1.3:** If  $t = 3$  the bound  $|\bar{e}| \leq t + 1$  holds for any negative edge  $\bar{e}$  of  $\overline{G}_S$  by Lemma 3.1, so we will assume that  $t \geq 4$ . Since none of the manifolds  $P \times S^1/[m]$  is hyperbolic (cf [5, Proposition 3.4]), by Corollary 3.4 the bound  $|\bar{e}| \leq t + 2$  holds for any negative edge  $\bar{e}$  of  $\overline{G}_S$ .

Suppose now there is an edge  $\bar{e}$  in  $\overline{G}_S$  with  $|\bar{e}| = t + 2$ , so that  $T$  is a positive surface and hence, by the parity rule, in  $G_S$  all edges are negative, so any cycle in  $G_S$  is even sided.

By [5, Lemma 2.2(b)], the reduced graph  $\overline{G}_S$  has a vertex  $u$  of degree  $n \leq 4$ . Counting endpoints of edges of  $G_S$  around  $v$  yields the relations

$$6t \leq \Delta \cdot t \leq n \cdot (t + 2),$$

which along with the restriction  $n \leq 4$  imply that  $n = t = 4$ ,  $\Delta = 6$ , and that each of the 4 edges  $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4$  incident to  $u$  in  $\overline{G}_S$  has size  $t + 2 = 6$ , as shown in Fig. 13. We orient the edges of  $\bar{e}_1, \bar{e}_2$  away from  $u$  as indicated in Fig. 13.

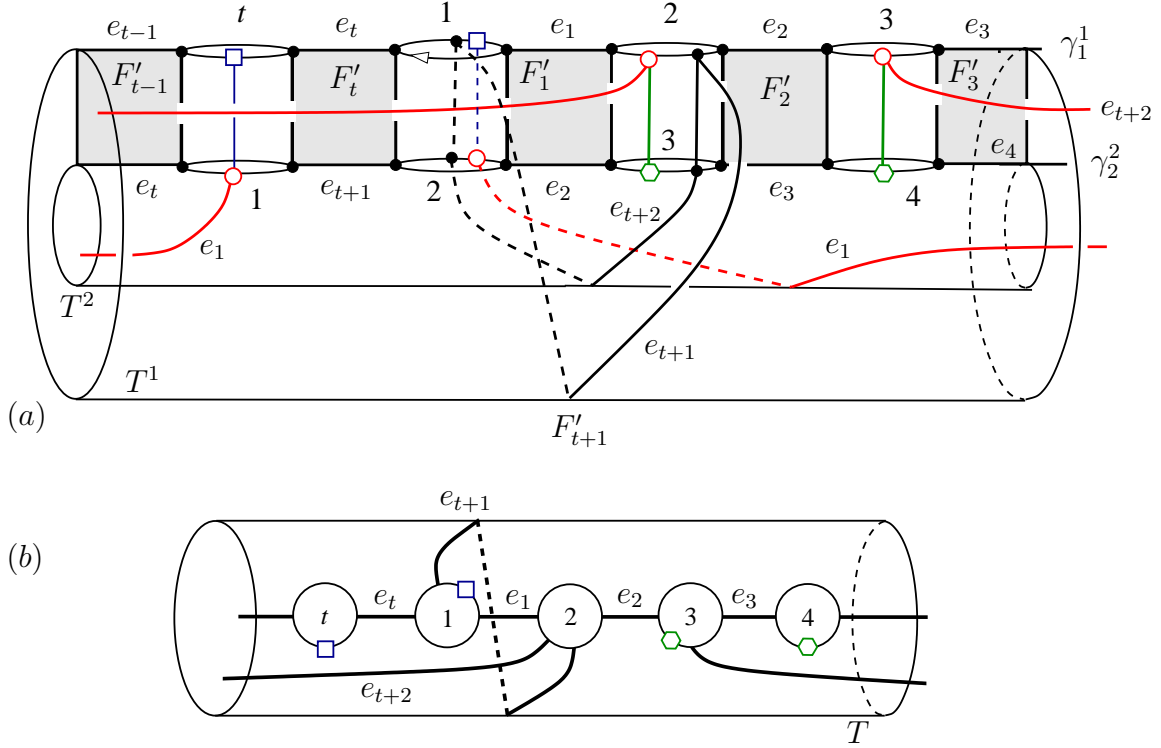


FIGURE 12.

By Lemma 3.2, reversing the orientation of the vertices of  $G_S$  and relabeling the vertices of  $G_T$ , if necessary, we may assume that the permutation induced by the oriented edge  $\bar{e}_1 = \{e_1, \dots, e_6\}$  around  $u$  is of the form  $x \mapsto x + 1$ , while the permutation induced by the oriented edge  $\bar{e}_2 = \{a_1, \dots, a_6\}$  is of the form  $x \mapsto x \pm 1$ .

We consider in detail the case where  $\bar{e}_2$  induces the permutation  $x \mapsto x - 1$ ; the case where this permutation is  $x \mapsto x + 1$  follows by a similar argument. We may assume that the endpoints of the edges in  $\bar{e}_1, \bar{e}_2$  are labeled as in Fig. 13. We may also assume that the edges of  $\bar{e}_1$  lie in  $T$  as shown in Fig. 14; this figure is obtained from the graph of Fig. 12(b) with  $t = 4$ , except here  $T$  is shown cut along the edges  $e_1, e_5$  of  $\bar{e}_1$ .

The edges  $a_2, a_6$  of  $\bar{e}_2$  are parallel in  $G_S$  and have endpoints on the vertices  $v_3, v_4$  of  $G_T$ , so by Lemma 2.1(b) for some  $i \in \{2, 6\}$  the edge  $a_i$  is not parallel in  $T$  to  $e_3$ ; it is not hard to see that there is only one embedding of  $a_i$  in  $T$ , as shown in Fig. 14.

Let  $F$  be the bigon of  $\bar{e}_2$  cobounded by  $a_{i-1}$  and  $a_i$ ; from the point of view of  $M_T$ , the edge  $a_{i-1}$  of  $F$  lies in  $T^1$  while the edge  $a_i$  lies in  $T^2$ . Since the edge  $a_i^2 \subset T^2$  has one endpoint in the interval  $(e_2^2, e_6^2) \subset v_3^2$  and its other endpoint in the interval  $(e_4^2, e_3^2) \subset v_4^2$ , represented by open circles in Fig. 14, by following the corners of  $F$  up along the strings  $I'_{2,3}$  and  $I'_{3,4}$  in Fig. 12(a) (with  $t = 4$ ) we can see that  $a_{i-1}^1 \subset T^1$  has one endpoint in the interval  $(e_1^1, e_5^1) \subset v_2^1$  and its other endpoint in the interval  $(e_3^1, e_2^1)$  of  $v_3^1$ . The possible locations of the endpoints of  $a_{i-1}$  around  $v_2$  and  $v_3$  are indicated by open squares in Fig. 14; this situation is



**3.3. The manifolds**  $(M_t, T_0)$ ,  $t \geq 4$ . For each  $t \geq 4$  let  $T$  be a  $t$ -punctured torus and  $M_t$  the manifold  $M_T/\psi$ , where  $M_T = T \times I$  is the solid handlebody with complete disk system  $F'_1, F'_2, \dots, F'_{t+1}$  of Fig. 12(a) considered in Case 2 of Lemma 3.3 and  $\psi : T^1 = T \times 0 \rightarrow T^2 = T \times 1$  is the homeomorphism uniquely determined up to isotopy by the conditions  $\psi(v_i^1) = v_i^2$  for  $1 \leq i \leq t$  and  $\psi(e_j^1) = e_j^2$  for  $1 \leq j \leq t+1$ . The basic properties of  $M_t$  are summarized in the next lemma.



**Lemma 3.5.** *The manifold  $M_t$  is orientable and  $\partial M_t$  is a torus  $T_0$ . Moreover, the pair  $(M_t, T_0)$  is irreducible and  $T = T^1/\psi$  is a properly embedded essential punctured torus in  $(M_t, T_0)$ .*

*Proof.* If  $T^1 \subset \partial M_T$  and  $T^2 \subset \partial M_T$  are oriented by the normal vectors  $\vec{N}^1, \vec{N}^2$  in Fig. 4 then the conditions  $\psi(v_i^1) = v_i^2$  for  $1 \leq i \leq t$  and  $\psi(e_j^1) = e_j^2$  for  $1 \leq j \leq t+1$  imply that the homeomorphism  $\psi : T^1 \rightarrow T^2$  is orientation preserving, hence  $M_t$  is orientable. Clearly,  $\partial M_t = (I'_{1,2} \sqcup I'_{2,3} \sqcup \cdots \sqcup I'_{t,1})/\psi$  is a single torus  $T_0$ , and hence  $T = T_1/\psi$  is a properly embedded torus in  $(M_t, T_0)$ .

If  $T = T_1/\psi$  compresses in  $M_t = M_T/\psi$  then, as  $M_T$  is obtained by cutting  $M_t$  along  $T$ , it follows that  $T^1$  or  $T^2$  compresses in  $M_T$ , which is not the case since  $M_T = T \times I$ . Therefore  $T$  is incompressible in  $M_t$ .

Since  $M_T$  is a handlebody, hence irreducible, it now follows that  $M_t$  is also irreducible. So, if  $T_0$  compresses in  $M_t$  then  $M_t$  must be a solid torus, contradicting the fact that  $T$  is incompressible in  $M_t$ . Therefore  $T_0$  is incompressible and  $T$  is essential in  $(M_t, T_0)$ .  $\square$

We now show that the bigons in the complete disk system of  $M_T$  give rise to a twice-punctured torus  $S$  embedded in  $(M_t, T_0)$ .

**Lemma 3.6.** *For each  $t \geq 4$  there is a separating, essential, twice-punctured torus  $S$  in  $(M_t, T_0)$  such that  $\Delta(\partial T, \partial S) = 3$  and  $G_S = S \cap T \subset S$ ,  $G_{T,S} = S \cap T \subset T$  are the graphs shown in Figs. 16 and 17.*

*Proof.* Let  $t \geq 4$ , and consider the faces  $F'_1, \dots, F'_{t+1}$  and the edges  $e_j^i$  for  $1 \leq j \leq t+2$ ,  $i = 1, 2$ , embedded in the handlebody  $M_T = T \times I$  and  $T^i$ , respectively, as shown in Fig. 12(a), with  $\psi(e_j^1) = e_j^2$  for all  $j$ . We add the following objects to  $M_T$  as indicated in Fig. 15:

- one bigon face parallel to  $F'_i$  for  $i = 2, t$ ,
- two bigon faces parallel to  $F'_i$  for  $3 \leq i \leq t-1$ ,
- one more edge parallel to each of the edges  $e_2^2, e_3^2$  and  $e_t^1, e_{t+1}^1$ .

Therefore, in  $T$ , the edges  $e_1, e_{t+2}$  get no parallel edges, each of the edges  $e_i$  for  $i = 2, t+1$  gets one parallel edge denoted by  $e'_i$ , and each of the edges  $e_i$  for  $3 \leq i \leq t$  gets two parallel edges denoted  $e'_i, e''_i$ . The new collections of edges  $e_i, e'_i, e''_i$  produce graphs  $G^i \subset T^i$  for  $i = 1, 2$  isomorphic to the graph in Fig. 16 such that, after a slight isotopy of the edges  $e'_i, e''_i$  if necessary, satisfy  $\psi(G^1) = G^2$ .

Now connect all the old and newly added edges of  $T^1, T^2$  via mutually disjoint corners as shown in Fig. 15. Observe that the 6-cycle  $\mathcal{C}$  in  $\partial M_T$  containing the edges  $e_1^2, (e'_{t+1})^1, (e''_3)^2, e_{t+2}^1, (e'_2)^2, (e''_t)^1$  is disjoint from the complete disk system  $F'_1, \dots, F'_{t+1}$  of  $M_T$ , hence  $\mathcal{C}$  bounds a 6-sided disk ‘face’  $F'_\mathcal{C}$  in  $M_T$  disjoint from all the other bigons in  $M_T$ .

In this way we obtain a collection  $\mathcal{F}$  of  $3t-2$  disjoint disk faces embedded in  $M_T$ :  $2(3) + 3(t-3) = 3t-3$  bigons and one 6-sided disk face. The condition  $\psi(G^1) = G^2$  guarantees that  $S = \mathcal{F}/\psi \subset (M, T_0)$  is a properly embedded surface which intersects  $T$  transversely in the graph  $G_{T,S} = T \cap S \subset T$  of Fig. 16. Moreover, the collection of corners of faces in  $\mathcal{F}$  whose

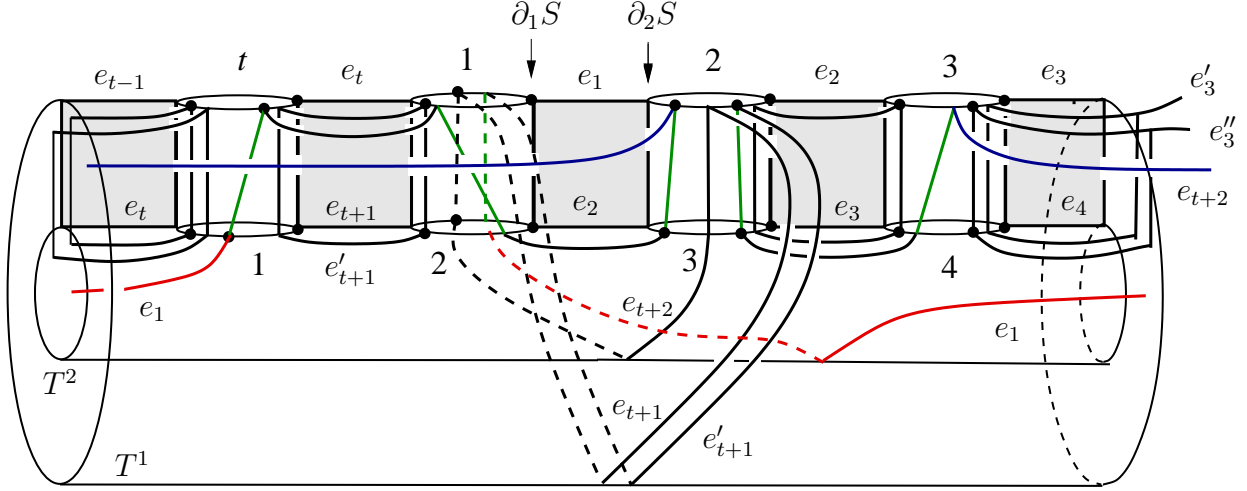


FIGURE 15.

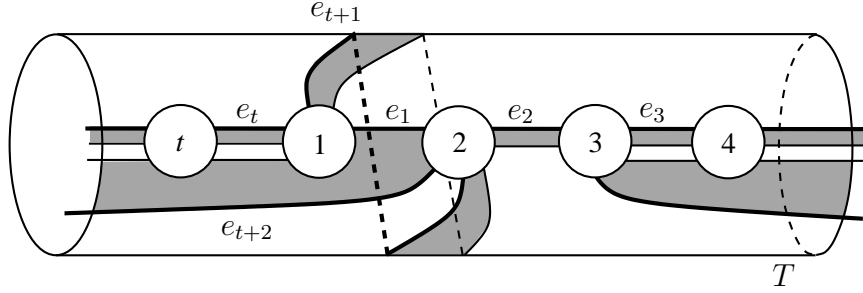


FIGURE 16.

endpoints are capped with a small closed disk in Fig. 15 form one boundary component  $\partial_1 S$  of  $S$ , while the remaining corners form a second boundary component  $\partial_2 S$ ; thus  $|\partial S| = 2$ .

Therefore all faces of the graph  $G_S = S \cap T \subset S$  are disks, and  $G_S$  has 2 vertices,  $3t$  edges, and  $|\mathcal{F}| = 3t - 2$  faces, so  $\hat{S}$  is a surface with Euler number 0; since each vertex of  $G_{T,S}$  has degree 6, it follows that  $\Delta(\partial S, \partial T) = 3$ .

Now, the faces of  $G_{T,S}$  can be colored black or white as shown in Fig. 16, and this coloring induces a corresponding black and white coloring of the components of  $M_T \setminus \cup \mathcal{F}$  such that each face in  $\mathcal{F}$  is adjacent to opposite colored components, which implies that  $S$  separates  $M_t$  and hence that  $S$  is a 2-punctured separating torus. We denote by  $S^B, S^W$  the closures of the components of  $M_t \setminus S$ .

The graph  $G_S$  can be determined by following the boundary circle  $\partial_1 S$  in Fig. 15 in the direction of increasing labels; starting at the endpoint of  $e_1^1$  in  $v_1 \subset \partial T$  we find that the collections of edges  $\{e_1, \dots, e_{t+2}\}$ ,  $\{e'_2, \dots, e'_{t+1}\}$ , and  $\{e''_3, \dots, e''_t\}$  form distinct parallelism

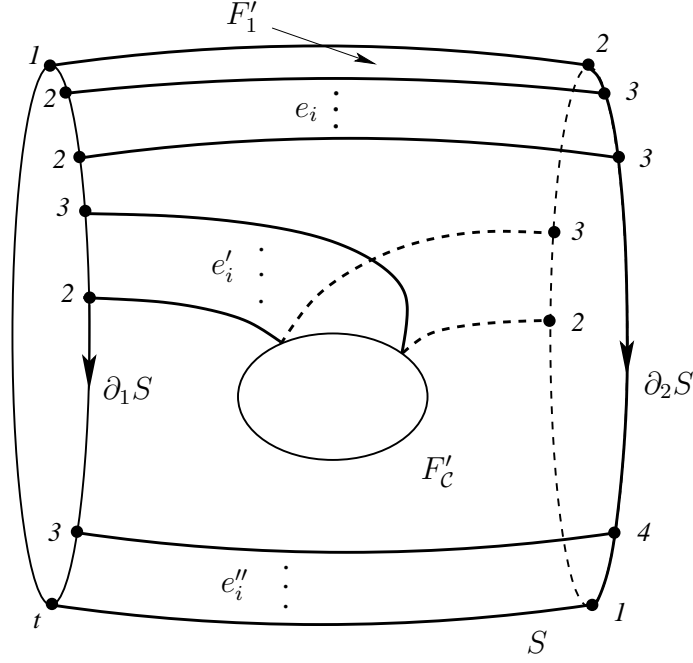


FIGURE 17.

classes in  $T$  and are read in this order as we traverse the circle  $\partial_1 S$  in Fig. 15, and so  $G_S$  must be the graph shown in Fig. 17.

For each  $* \in \{B, W\}$  the manifold  $S^*$  is irreducible with  $\partial S^*$  a genus two surface. Notice that  $G_{T,S}$  contains Scharlemann cycle disk faces in  $S^*$  of distinct lengths; any such Scharlemann cycle disk face is nonseparating in  $S^*$ , whence  $S^*$  is a genus two handlebody with complete disk system any pair of Scharlemann cycle disk faces of  $G_{T,S}$  in  $S^*$  of different sizes.

For  $S^B$  we can take as complete disk system the bigon  $x$  of  $G_{T,S}$  containing the edge  $e_2$  and the  $t$ -sided face  $y$  containing  $e_{t+2}$ , so that  $\pi_1(S^B) = \{x, y \mid -\}$  and  $\partial_1 S \subset \partial S^B$  is represented by the word in  $\pi_1(S^B)$  obtained by reading the consecutive intersections of  $\partial_1 S$  with the disks  $x$  and  $y$ . Following  $\partial_1 S$  around in Fig. 15, we can see that  $\partial_1 S = (yx)^2 y^{t-2}$ , which is not a primitive word in  $\pi_1(S^B) = \{x, y \mid -\}$ ; therefore  $S$  is incompressible in  $S^B$  by [4, Lemma 5.2]. Similarly, in  $S^W$  we can take as complete disk system the white bigon  $X$  in  $G_{T,S}$  with corners along  $v_3$  and  $v_4$  and the white  $t+4$ -sided face  $Y$  containing  $e_{t+2}$ ; we then compute that  $\partial_1 S \subset \partial S^W$  is represented by the word  $Y^{t+3}XYX$  in  $\pi_1(S^W) = \{X, Y \mid -\}$ , which is not primitive, so  $S$  is incompressible in  $S^W$  too. Therefore  $S$  is incompressible, hence essential, in  $M_t$ .  $\square$

We now complete the proofs of the remaining main results of this paper.

**Proof of Proposition 1.1:** Parts (1), (2) and (3) follow immediately from Lemmas 3.5 and 3.6. By Lemma 2.2, the manifold  $M_t(\partial T)$  is a torus bundle over a circle with fiber  $\widehat{T}$ . Finally, by [5, Proposition 3.4](c), if  $(P \times S^1/[m], T_0)$  contains two  $\mathcal{K}$ -incompressible tori  $T, T'$

then  $\Delta(T, T') \in \{0, 1, 2, 4\}$ ; since  $M_t$  contains the essential torus  $S$  with  $\Delta(\partial T, \partial S) = 3$ , it follows that  $M_t$  is not homeomorphic to any of the manifolds  $P \times S^1/[m]$ , so part (4) holds.  $\square$

**Proof of Proposition 1.2:** That  $T$  is positive follows from Lemma 2.1(c). Suppose  $(M, T_0)$  does not satisfy parts (1) and (3) of the proposition. By Lemmas 2.3(1) and 3.1 we then have that  $t \geq 4$ , and so by Lemmas 2.3(2), 3.2, 3.3, and the definition of  $M_t$ , we have that  $(M, T_0)$  is homeomorphic to  $(M_t, T_0)$ . Since, by Proposition 1.1,  $M_t$  is not a manifold of the form  $P \times I/[m]$ , the bound  $|\bar{e}| \leq t+2$  holds for all negative edges  $\bar{e}$  of  $\overline{G}_R$  by Corollary 3.4. Therefore part (2) holds.  $\square$

**Proof of Lemma 1.4:** Assume parts (1), (2) and (4) of the lemma do not hold; then  $t \geq 4$  and  $(M, T_0) \approx (M_t, T_0)$  by Proposition 1.2(2), and there is a negative edge in  $\overline{G}_Q$  of length  $|\bar{e}| \geq t+2$ . By Corollary 3.4 we then have that  $|\bar{e}| = t+2$ , so the lemma holds.  $\square$

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TEXAS AT EL PASO, EL PASO, TX 79968, USA

*E-mail address:* valdez@math.utep.edu